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粘性流体中の渦輪の運動

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1 Introduction

The motion of a vortex ring is one of the most classical and fundamental problems of vortex dynamics. Extending Kelvin's result, Dyson (1893) obtained the speed U of an axisymmetric vortex ring, embedded in an inviscid incompressible fluid, up to third (virtually fourth) order in a small parameter:

$$U=\frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8}{\epsilon}\right)-\frac{1}{4}-\frac{3\epsilon^2}{8}\left[\log\left(\frac{8}{\epsilon}\right)-\frac{5}{4}\right]+O(\epsilon^4\log\epsilon)\right\}, \tag{1}$$

where Γ is the circulation, R_0 is the ring radius and $\epsilon=\delta/R_0$ is the radius ratio of core δ to ring (see also Fraenkel 1972). The vorticity is the simplest one proportional to the distance from the symmetry axis. We count Kelvin's formula as the first order and ϵ^2-term as the third. The vortex ring induces a local straining field on itself which deforms the core into an ellipse at second order:

$$r=\delta\left\{1+\epsilon^2\left[\frac{3}{8}\log\left(\frac{8}{\epsilon}\right)-\frac{17}{32}\right]\cos 2\theta+\cdots\right\}. \tag{2}$$

where (r,θ) are local cylindrical coordinates about the core center which will be introduced in §2. It is remarkable that the inclusion of the third-order term in the propagation velocity achieves a great improvement in approximation. In reality, (1) exhibits fair agreement even with the exact value for the fat limit of Hill's spherical vortex ($\epsilon=\sqrt{2}$).

The viscosity acts to diffuse the vorticity. Its influence on the propagation speed, at large Reynolds number, was calculated by Tung and Ting (1967) (Callegari and Ting 1978) and Saffman (1970), up to $O((\nu/\Gamma)^{1/2})$, as

$$U=\frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8R_0}{2\sqrt{\nu t}}\right)-\frac{1}{2}(1-\gamma+\log 2)+\cdots\right\}, \tag{3}$$

where ν is the kinematic viscosity of fluid, t is the time, and $\gamma = 0.57721566\cdots$ is Euler's constant. The vorticity distribution has a decaying Gaussian profile with circular symmetry.

Recent direct numerical simulations of fully developed turbulence have unveiled that the small-scale structure is dominated by high-vorticity regions concentrated in tubes (see, for example, Siggia 1981; Kerr 1985; Hosokawa and Yamamoto 1989). They occupy a relatively small fraction of the total volume, but are responsible for a much larger fraction of viscous dissipation. This observation reminds us of Townsend's idea that concentrated vortices are looked upon as sinews of turbulence. Inspired by this spirit, Moffatt *et al.* (1994, 1996) developed a large-Reynolds-number asymptotic theory to solve Navier-Stokes equations for a columnar vortex subjected to uniform non-axisymmetric irrotational strain. The solution is universal in that it satisfactorily accounts for the viscous structure such as dissipation field obtained by numerical computation (Kida and Ohkitani 1992). The viscosity is an agent to pick out vorticity distribution. At leading order, the Burgers vortex is obtained, and at the next order ($O(\nu/\Gamma)$), a quadrupole component emerges, reflecting an elliptical vorticity distribution. The salient feature is that major axis of the ellipse is aligned at 45° to the principal axis of the external strain. This fact leads us to the belief that the strained crosssection of a propagating vortex ring, commonly observed in nature, is established as an equilibrium of viscous elliptic core opposing a self-induced strain.

The aim of our study is to elucidate the structure of strained core and its influence on the translation speed of an axisymmetric vortex ring. As a first step, we present, in this paper, a general framework to address it. A partial answer is given as to how viscosity affects the radial drift of vorticity.

The method of matched asymptotic expansions has been developed to derive the velocity of a slender curved vortex tube in a fluid with and without viscosity (Tung and Ting 1967; Widnall *et al.* 1971; Callegari and Ting 1978; Klein and Majda 1991; Ting and Klein 1991). However it is limited to the second-order curvature effect (Moore and Saffman 1972; Fukumoto and Miyazaki 1991). The self-induced field of a vortex ring makes its appearance at second order in $\epsilon = (\nu/\Gamma)^{1/2}$, and the translation speed is affected at the next order. We make an attempt to extend asymptotic expansions to a higher order and to calculate the speed of a vortex ring up to $0(\epsilon^3)$.

The existing asymptotic formula of the potential flow caused by a circular vortex loop is not sufficient to carry through this program. After a brief statement about the general setting of asymptotic expansions in §2, we manipulate an asymptotic formula of the Biot-Savart integral accommodating an arbitrary vorticity distribution in § 3. In §4, the inner expansions are recalled and extended to second order. Based on them, we establish, in § 5, a general formula of the translation velocity of a vortex ring, valid up to third order. Dyson's formula (1) is restored in a special case. Moreover, It is revealed that the radius of the loop consisting of the stagnation points in the core, when viewed from the frame moving with the core, expands linearly in time owing to the action of viscosity. Our procedure pursuing higher-order asymptotics spotlights the significance of the dipole distributed along the ring. Its strength must be prescribed at the initial instant and thereby the problem of undetermined constants at $O(\epsilon)$ is remedied.

2 Formulation of the matched asymptotic expansions

Two length scales are available, namely, typical scales of the core radius δ and the ring radius R_0 . We assume that their ratio is very small. We retain only the slow mode of core dynamics, suppressing fast waves on the core. Then, in view of (1), time scale is of order $R_0/(\Gamma/R_0) = R_0^2/\Gamma$. In the presence of viscosity, the core radius grows as $\delta \sim (\nu t)^{1/2} \sim (\nu/\Gamma)^{1/2} R_0$ during this time. Thus our assumption reads

$$\epsilon = \frac{\delta}{R_0} = \sqrt{\frac{\nu}{\Gamma}} \ll 1 . \tag{4}$$

Let us introduce cylindrical coordinates (ρ, ϕ, z) with z-axis along the axis of symmetry and ϕ along the vortex lines. The vorticity distribution $\boldsymbol{\omega}$ is axisymmetric but otherwise arbitrary:

$$\boldsymbol{\omega} = \zeta(\rho, z)\boldsymbol{e}_\phi , \tag{5}$$

where $\boldsymbol{e}_\phi$ is the unit vector in the azimuthal direction. The Stokes streamfunction ψ for the flow produced by (5) is written down at once:

$$\psi = -\frac{\rho}{4\pi}\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} \frac{\zeta(\rho', z')\cos\phi' d\rho' d\phi' dz'}{\sqrt{\rho^2 - 2\rho\rho'\cos\phi' + \rho'^2 + (z-z')^2}} . \tag{6}$$

As is well known, the asymptotic expression of (6) for an infinitely thin core is not convergent near the core center. A way out is to connect it to the viscous flow which decays rapidly within the core, in an analogous way as the boundary layer. Thus we are led to the inner and outer expansions (Tung and Ting 1967). The inner region has length scale of the core radius δ and there we seek the solution of the Navier-Stokes equations matched to the outer solution given by (6).

It is expedient to choose a coordinate frame moving with the core center $(R(t), Z(t))$ in which we introduce local cylindrical coordinates (r, θ) such that

$$\rho = R(t) + r\cos\theta , \qquad z = Z(t) + r\sin\theta . \tag{7}$$

Introduce dimensionless variables:

$$\left.\begin{aligned} r^* = r/\epsilon R_0\,, \quad t^* = t/\tfrac{R_0}{\Gamma}\,, \quad \psi^* = \tfrac{\psi}{\Gamma R_0}\,, \quad \zeta^* = \zeta/\tfrac{\Gamma}{R_0^2\epsilon^2}\,, \\ v^* = v/\tfrac{\Gamma}{R_0\epsilon}\,, \quad (\dot{R}^*, \dot{Z}^*) = (\dot{R}, \dot{Z})/\tfrac{\Gamma}{R_0}\,. \end{aligned}\right\} \tag{8}$$

Here v is the velocity relative to the moving coordinates and the distinction in normalization between the last two of (8) is to be kept in view. The equations handled in the inner region is the vorticity equation, combined with the relation between ζ and ψ. Dropping the stars, they take the following form:

$$\begin{aligned} &\frac{\partial \zeta}{\partial t} + \frac{1}{\epsilon^2}\left(u\frac{\partial \zeta}{\partial r} + \frac{v}{r}\frac{\partial \zeta}{\partial \theta}\right) - \frac{1}{\epsilon\rho^2}\left(\frac{\partial \psi}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial \zeta}{\partial \theta}\cos\theta\right) \\ = \;& \Delta\zeta + \frac{\epsilon}{\rho}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\zeta - \frac{\epsilon^2}{\rho^2}\zeta\,, \end{aligned} \tag{9}$$

$$\zeta = \frac{1}{\rho}\Delta\psi - \frac{\epsilon}{\rho^2}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\psi\,. \tag{10}$$

where

$$\rho = R + \epsilon r\cos\theta\,, \tag{11}$$

Δ is the two-dimensional Laplacian,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,, \tag{12}$$

and u and v are the r- and θ-components of the relative velocity $\boldsymbol{v}$:

$$u = \frac{1}{r\rho}\frac{\partial \psi}{\partial \theta} - \epsilon(\dot{Z}\sin\theta + \dot{R}\cos\theta)\,, \tag{13a}$$

$$v = -\frac{1}{\rho}\frac{\partial \psi}{\partial r} - \epsilon(\dot{Z}\cos\theta - \dot{R}\sin\theta)\,. \tag{13b}$$

We look for the following form of the solution of (9) and (10):

$$\zeta = \zeta^{(0)} + \epsilon\zeta^{(1)} + \epsilon^2\zeta^{(2)} + \epsilon^3\zeta^{(3)} + \cdots\,, \tag{14a}$$

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + \epsilon^3\psi^{(3)} + \cdots\,, \tag{14b}$$

$$R = R^{(0)} + \epsilon R^{(1)} + \epsilon^2 R^{(2)} + \cdots\,, \tag{14c}$$

$$Z = Z^{(0)} + \epsilon Z^{(1)} + \epsilon^2 Z^{(2)} + \cdots\,, \tag{14d}$$

Here $\zeta^{(i)}$ and $\psi^{(i)}$ $(i = 0, 1, 2, 3, \cdots)$ are taken to be as functions of r, θ, and t.

The permissible solution must satisfy*,

$$u \text{ and } v \text{ are finite at } r = 0\,, \tag{15}$$

and the requirement that it smoothly matches the outer solution singles out the values of $\dot{R}^{(i)}$ and $\dot{Z}^{(i)}$ $(i = 0, 1, 2, \cdots)$.

*This condition is better than the restrictive one that $u = v = 0$ at $r = 0$.

3 Outer solution

The streamfunction ψ_m for the flow induced by a circular vortex loop $\zeta = \delta(r-R)\delta(z-Z)$ of unit strength is obtainable from (6):

$$\begin{aligned}\psi_m(\rho,z,R) &= -\frac{\rho}{4\pi}\int_0^{2\pi}\frac{R\cos\phi' d\phi'}{\sqrt{\rho^2-2\rho R\cos\phi'+R^2+z^2}} \\ &= -\frac{1}{2\pi}\left[K\left(\frac{r_2-r}{r_2+r}\right)-E\left(\frac{r_2-r}{r_2+r}\right)\right],\end{aligned} \tag{16}$$

where z is redefined relative to Z, $r_2 = (4R^2+r^2+4Rr\cos\theta)^{1/2}$ is the longest distance from the point (ρ, z) to the loop, and K and E are the complete elliptic integrals of the first and second kinds, respectively, with $(r_2-r)/(r_2+r)$ being the modulus. We call (16) the monopole field. So far, this has been exclusively employed as the outer solution.

It turns out however that, when going into higher orders, (16) is not enough to be qualified as the outer solution. The elaboration of the detailed structure of (6) is unavoidable. To this aim, it is advantageous to adapt Dyson's technique to an arbitrary distribution of vorticity:

$$\begin{aligned}\psi &= -\frac{\rho}{4\pi}\iint dx'dz'\zeta(x',z')e^{x'\frac{\partial}{\partial R}-z'\frac{\partial}{\partial z}}\int_0^{2\pi}\frac{R\cos\phi' d\phi'}{\sqrt{\rho^2-2\rho R\cos\phi'+R^2+z^2}} \\ &= \iint dx'dz'\zeta(x',z')\left\{1+\frac{\partial}{\partial R}-z'\frac{\partial}{\partial z}+\frac{1}{2!}\left(\frac{\partial}{\partial R}-z'\frac{\partial}{\partial z}\right)^2+\frac{1}{3!}\left(\frac{\partial}{\partial R}-z'\frac{\partial}{\partial z}\right)^3\right. \\ &+ \left.\frac{1}{4!}\left(\frac{\partial}{\partial R}-z'\frac{\partial}{\partial z}\right)^4+\frac{1}{5!}\left(\frac{\partial}{\partial R}-z'\frac{\partial}{\partial z}\right)^5+\frac{1}{6!}\left(\frac{\partial}{\partial R}-z'\frac{\partial}{\partial z}\right)^6+\cdots\right\}\psi_m.\end{aligned} \tag{17}$$

The expected spatial dependence of vorticity distribution is

$$\zeta^{(0)} = \zeta^{(0)}, \tag{18a}$$
$$\zeta^{(1)} = \zeta_{11}^{(1)}\cos\theta, \tag{18b}$$
$$\zeta^{(2)} = \zeta_0^{(2)}+\zeta_{21}^{(2)}\cos 2\theta, \tag{18c}$$
$$\zeta^{(3)} = \zeta_{11}^{(3)}\cos\theta+\zeta_{12}^{(3)}\sin\theta+\zeta_{31}^{(3)}\cos 3\theta. \tag{18d}$$

For $\zeta_{jk}^{(i)}$, i denotes the order of perturbations and j the Fourier mode, with $k=1$ and 2 being corresponding to $\cos j\theta$ and $\sin j\theta$ respectively. Plugging (18a)–(18d) into (17), implementing the integration with respect to x' and z', and thereafter taking the derivatives of ψ_m, substituted from (16), with respect to R and z, we gain the asymptotic form of the outer solution valid at $r \ll R$, which is expressed in dimensioless form as

$$\psi = -\frac{R}{2\pi}\Gamma\log\left(\frac{8R}{\epsilon r}\right)+\epsilon\left(-\frac{\Gamma}{4\pi}\left[\log\left(\frac{8R}{\epsilon r}\right)-1\right]r\cos\theta+d\frac{\cos\theta}{r}\right)$$

$$
\begin{aligned}
&+\epsilon^2\left(-\frac{\Gamma}{2^5\pi R}\left\{\left[2\log\left(\frac{8R}{\epsilon r}\right)+1\right]r^2-\left[\log\left(\frac{8R}{\epsilon r}\right)-2\right]r^2\cos\theta\right\}-\frac{R}{2\pi}\Gamma^{(2)}\right.\\
&\left.+\frac{d}{2R}\left[\log\left(\frac{8R}{\epsilon r}\right)+\frac{\cos 2\theta}{2}\right]+q\frac{\cos 2\theta}{r^2}\right)\\
&+\epsilon^3\left(\frac{3\Gamma}{2^7\pi R^2}\left\{\left[\log\left(\frac{8R}{\epsilon r}\right)-\frac{1}{3}\right]r^3\cos\theta-\left[\log\left(\frac{8R}{\epsilon r}\right)-\frac{7}{3}\right]r\cos 3\theta\right\}\right.\\
&-\frac{\Gamma^{(2)}}{4\pi}\left[\log\left(\frac{8R}{\epsilon r}\right)-1\right]r\cos\theta-\frac{d}{8R^2}\left\{\left[\log\left(\frac{8R}{\epsilon r}\right)-\frac{7}{4}\right]r\cos\theta+\frac{r\cos 3\theta}{4}\right\}\\
&-\frac{1}{2\pi}\left\{\frac{1}{4}\Big[2\pi\int_0^\infty r^3\zeta_0^{(2)}dr\Big]+R\Big[\pi\int_0^\infty r^2\zeta_{11}^{(3)}dr\Big]+\frac{1}{4}\Big[\pi\int_0^\infty r^3\zeta_{21}^{(2)}dr\Big]\right\}\frac{\cos\theta}{r}\\
&+\frac{q}{4R}\left(\frac{\cos\theta}{r}+\frac{\cos 3\theta}{r}\right)-\frac{1}{\pi R}\left\{\frac{1}{3\cdot 2^8}\Big[2\pi\int_0^\infty r^5\zeta^{(0)}dr\Big]-\frac{R}{8\cdot 4!}\Big[\pi\int_0^\infty r^6\zeta_{11}^{(1)}dr\Big]\right.\\
&\left.\left.+\frac{R^2}{4!}\Big[\pi\int_0^\infty r^5\zeta_{21}^{(2)}dr\Big]+\frac{R^3}{6}\Big[\pi\int_0^\infty r^4\zeta_{31}^{(3)}dr\Big]\right\}\frac{\cos 3\theta}{r^3}\right)+\cdots, \qquad (19)
\end{aligned}
$$

where

$$\Gamma = 2\pi\int_0^\infty r\zeta^{(0)}dr\,, \qquad (20)$$

$$\Gamma^{(2)} = 2\pi\int_0^\infty r\zeta_0^{(2)}dr\,, \qquad (21)$$

and $\Gamma = 1$ when nondimensionalised, and d and q are tied with the strength of low-order dipole and quadrupole:

$$
\begin{aligned}
d &= -\frac{1}{2\pi}\left\{\frac{1}{4}\Big[2\pi\int_0^\infty r^3\zeta^{(0)}dr\Big]+R\Big[\pi\int_0^\infty r^2\zeta_{11}^{(1)}dr\Big]\right\}, \qquad (22)\\
q &= -\frac{1}{2\pi R}\left\{-\frac{1}{2^6}\Big[2\pi\int_0^\infty r^5\zeta^{(0)}dr\Big]+\frac{R}{8}\Big[\pi\int_0^\infty r^4\zeta_{11}^{(1)}dr\Big]+\frac{R^2}{2}\Big[\pi\int_0^\infty r^3\zeta_{21}^{(2)}dr\Big]\right\}. \qquad (23)
\end{aligned}
$$

The terms multiplied by Γ stem from $\Gamma\psi_m$, which are augmented by the induction velocities due to the dipole, quadrupole, hexapole distributed along the center $r = 0$ of the core. Parts of (19) supply the matching conditions on the inner solution. The distributions of $\zeta_{11}^{(1)}$, $\zeta_0^{(2)}$, $\zeta_{11}^{(3)}$, $\zeta_{12}^{(3)}$, $\zeta_{21}^{(2)}$, and $\zeta_{31}^{(3)}$ are yet unknown, but are fixed by the inner expansions and the matching procedure. It will be clarified that the dipole components $\zeta_{11}^{(1)}$, $\zeta_{11}^{(3)}$, $\zeta_{12}^{(3)}$ are distinctive and that the initial condition is necessary to place the constraints on them. In the subsequent sections we inquire into the flow field inside the core.

4 Inner expansions up to second order

Before going to third order, we give a brief outline of the inner perturbations up to second order.

Collecting like powers of ϵ in (9) and (10), along with (11)–(13b), substituted from (14a)–(14d), the Navier-Stokes equations at each order are deduced successively.

At $O(\epsilon^0)$, we obtain the Jacobian form of the Euler equation:

$$[\zeta^{(0)}, \psi^{(0)}] = 0\,, \tag{24}$$

where we have defined as $[\zeta^{(0)}, \psi^{(0)}] = \partial(\zeta^{(0)}, \psi^{(0)})/\partial(r, \theta)$. Hence $\zeta^{(0)} = \mathcal{F}(\psi^{(0)})$ for some function $\mathcal{F}$. Suppose that the flow $\psi^{(0)}$ has a single stagnation point at $r = 0$, the streamlines being all closed around that point. Then it is probable that the solution of (24), coupled with $\zeta^{(0)} = \Delta\psi^{(0)}/R^{(0)}$ (see (10)), is radial $\psi^{(0)} = \psi^{(0)}(r)$, that is, the streamlines are circles (Moffatt *et al.* 1994)†. The functional form of $\psi^{(0)}(r)$ and $\zeta^{(0)}(r)$ remain undetermined at this level of approximation, but is determined through the axisymmetric part of the vorticity equation at $O(\epsilon^2)$:

$$\frac{\partial \zeta^{(0)}}{\partial t} = \left(\zeta^{(0)} + \frac{r}{2}\frac{\partial \zeta^{(0)}}{\partial r}\right)\frac{\dot{R}^{(0)}}{R^{(0)}} + \left(\frac{\partial^2 \zeta^{(0)}}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta^{(0)}}{\partial r}\right)\,. \tag{25}$$

where a dot stands for the differentiation with respect to time. We focus our attention to the case that, at the initial instant, the vorticity is concentrated in the circle of radius R_0:

$$\zeta^{(0)} = \delta(r - R_0)\delta(z) \quad \text{at} \quad t = 0\,. \tag{26}$$

Anticipating that $R^{(0)}$ is constant (see (34)), we obtain the decaying circular vortex

$$\zeta^{(0)} = \frac{1}{4\pi t}e^{-\frac{r^2}{4t}}\,. \tag{27}$$

(Tung and Ting 1967; Jiménez *et al.* 1996). Interestingly, the viscosity plays the role of choosing the distribution of vorticity, even in the limit of $\nu \to 0$.

The first-order perturbation $\psi^{(1)}$ obeys

$$\Delta\psi^{(1)} - a\psi^{(1)} = -\cos\theta v^{(0)} + aR_0 r(\dot{Z}^{(0)}\cos\theta - \dot{R}^{(0)}\sin\theta) + 2r\zeta^{(0)}\cos\theta\,, \tag{28}$$

where $R_0 = R^{(0)}$ with abuse of notation and

$$v^{(0)} = -\frac{1}{R_0}\frac{\partial \psi^{(0)}}{\partial r}\,, \tag{29}$$

$$a = -\frac{1}{v^{(0)}}\frac{\partial \zeta^{(0)}}{\partial r}\,. \tag{30}$$

Here we have used the fact that the axisymmetric part of $\zeta^{(1)}$ is suppressed from the result of (34) and the analysis of the vorticity equation at $O(\epsilon^3)$. The solution meeting the condition that the relative velocity $(u^{(1)}, v^{(1)})$ is finite at $r = 0$ is

$$\psi^{(1)} = \psi_{11}^{(1)}\cos\theta + \psi_{12}^{(1)}\sin\theta\,; \tag{31a}$$

†This result may be derived from the theorem proved by Gidas *et al.* (1979).

where

$$\begin{aligned}
\psi_{11}^{(1)} &= \tilde{\psi}_{11}^{(1)} - R_0 r \dot{Z}^{(0)}, && (31b)\\
&\quad \text{with } \tilde{\psi}_{11}^{(1)} = \Psi_{11}^{(1)} + c_{11}^{(1)} v^{(0)}, && (31c)\\
\psi_{12}^{(1)} &= c_{12}^{(1)} v^{(0)}, && (31d)
\end{aligned}$$

$c_{11}^{(1)}$ and $c_{12}^{(1)}$ are some constants, and $\Psi_{11}^{(1)}$ is a particular solution:

$$\Psi_{11}^{(1)} = v^{(0)} \int_0^r \frac{dr'}{r'[v^{(0)}(r')]^2} \left\{ \int_0^{r'} \eta v^{(0)}(\eta)[-v^{(0)}(\eta) + 2\eta\zeta^{(0)}(\eta)]d\eta \right\}, \qquad (32)$$

(Widnall *et al.* 1971; Callegari and Ting 1978).

Irrespective of any choice of the parameter values $c_{11}^{(1)}$ and $c_{12}^{(1)}$, the matching condition

$$\psi^{(1)} \sim -\frac{1}{4\pi}\left[\log\left(\frac{8R_0}{\epsilon r}\right) - 1\right] r\cos\theta \quad \text{as } r \to \infty, \qquad (33)$$

results in (3) and

$$\dot{R}^{(0)} = 0. \qquad (34)$$

To have an idea on the constants, we revisit the discrete model in an inviscid flow studied by Dyson. At leading order, it is the Rankine vortex, that is, the vorticity is constant in the circular core of unit length surrounded by an irrotational flow:

$$\zeta^{(0)} = \begin{cases} 1/\pi, \\ 0, \end{cases} \qquad v^{(0)} = \begin{cases} -r/2\pi, & (r \le 0) \\ -1/2\pi r, & (r > 0) \end{cases} \qquad (35)$$

The continuity of velocity across the core boundary $r = 1$ chooses

$$c_{11}^{(1)} = 5/8, \quad c_{12}^{(1)} = 0. \qquad (36)$$

(Widnall *et al.* 1971). However a difficulty arises when the discrete distribution is replaced by a continuous one, because the continuity condition is no longer of help. To make matters worse, both $c_{11}^{(1)}$ and $c_{12}^{(1)}$ admit arbitrary time dependence as long as we stick to the matching condition (33). This is true also for the discrete model, and therefore (36) is merely one possibility.

We can show that $c_{11}^{(1)}$ and $c_{12}^{(1)}$ serve as the parameters placing the circular core in the moving frame, to the accuracy of $O(\epsilon)$ in terms of the inner spatial scale. Increase of $c_{11}^{(1)}$ and $c_{12}^{(1)}$ by c amounts to the shift of the core-center by $\epsilon c/R_0$ in the ρ- and z-directions respectively. Without loss of generality, we may assume that $c_{12}^{(1)} = 0$. Still, a freedom of the choice of the location of the center in the radial direction is at our disposal. We realise that fixing the initial location of the core is equivalent to giving the value of d_0 at $t = 0$, and (33) is superseded by

$$\psi^{(1)} \sim \left\{ -\frac{\Gamma}{4\pi}\left[\log\left(\frac{8R_0}{\epsilon r}\right) - 1\right] r + \frac{d_0(t)}{r} \right\} \cos\theta \quad \text{as } r \to \infty. \qquad (37)$$

Comparison of (37) with (19) gives rise to the following identity:

$$d_0 = -\frac{1}{2\pi}\left\{\left[2\pi\int_0^\infty r^3\zeta^{(0)}dr\right] + R_0\left[\pi\int_0^\infty r^2\zeta_{11}^{(1)}dr\right]\right\}. \tag{38}$$

With the specification of $d_0(0)$, a proper formulation of the initial-value problem is completed. Yet, we suffer from arbitrariness of the temporal evolution of $d_0(t)$. We can verify that this is consistently absorbed into the third-order radial velocity $\dot{R}^{(2)}$ as exemplified at the end of §5. It implies that the perturbation solution is unique, while it has an infinite variety of representations.

Next, we proceed to the second-order perturbation $\psi^{(2)}$. It is shown to have the following θ-dependence:

$$\psi^{(2)} = \psi_0^{(2)} + \psi_{21}^{(2)}\cos\theta\,, \tag{39}$$

meaning that quadrupole is produced in conjunction with the elliptical core deformation. The governing equations and matching conditions are

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\psi_0^{(2)} = R_0\zeta_0^{(2)} + \frac{ra}{2R_0}\tilde{\psi}_{11}^{(1)} + \frac{1}{2R_0}\left[rv^{(0)} + r^2\zeta^{(0)} + \frac{\partial\psi_{11}^{(1)}}{\partial r} + \frac{\psi_{11}^{(1)}}{r}\right], \tag{40}$$

with

$$\psi_0^{(2)} \sim -\frac{1}{2^5\pi R_0}\left[2\log\left(\frac{8R_0}{\epsilon r}\right) + 1\right]r^2 + \frac{d_0}{2R_0}\log\left(\frac{8R_0}{\epsilon r}\right) \quad \text{as } r\to\infty\,, \tag{41}$$

and

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2} - a\right)\psi_{21}^{(2)} = \frac{b}{4R_0}\left[\tilde{\psi}_{11}^{(1)}\right]^2 + \frac{r^2a}{4}\dot{Z}^{(0)} + \frac{ra}{R_0}\tilde{\psi}_{11}^{(1)} + \frac{1}{2R_0}\left[rv^{(0)} + r^2\zeta^{(0)} + \frac{\partial\psi_{11}^{(1)}}{\partial r} - \frac{\psi_{11}^{(1)}}{r}\right] \tag{42}$$

with

$$\psi_{21}^{(2)} \sim \frac{1}{2^5\pi R_0}\left[\log\left(\frac{8R_0}{\epsilon r}\right) - 2\right]r^2 + \frac{d_0}{4R_0} \quad \text{as } r\to\infty\,, \tag{43}$$

where $\zeta_0^{(2)}$ is the axisymmetric part of the second-order vorticity perturbation and

$$b = -\frac{1}{v^{(0)}}\frac{\partial a}{\partial r}. \tag{44}$$

Finding $\zeta_0^{(2)}$ requests us to make headway to the vorticity equation at $O(\epsilon^4)$. It deserves mention that (42) is a natural generalisation of the quadrupole equation encountered by Moffatt *et al.* (1994, 1996) for a non-axisymmetric Burgers vortex.

Once the streamfunctions are available, the vorticity distribution is calculable through the formulae:

$$\zeta^{(1)} = \frac{1}{R_0}\left[a\tilde{\psi}_{11}^{(1)} + r\zeta^{(0)}\right]\cos\theta\,, \tag{45}$$

$$\zeta^{(2)} = \zeta_0^{(2)} + \left\{\frac{a}{R_0}\tilde{\psi}_{21}^{(2)} + \frac{b}{4R_0^2}\left[\tilde{\psi}_{11}^{(1)}\right]^2 + \frac{ra}{2R_0^2}\tilde{\psi}_{11}^{(1)}\right\}\cos 2\theta\,, \tag{46}$$

where

$$\tilde{\psi}_{21}^{(2)} = \psi_{21}^{(2)} + \frac{r^2}{4}\dot{Z}^{(0)}\,. \tag{47}$$

5 Third-order velocity of a vortex ring

At this stage, we are prepared to tackle the third-order problem. The dipole field again shows up as the result of nonlinear interactions among the mono-, di- and quadru-poles up to $O(\epsilon^2)$. It is this field that takes part in the correction to the ring speed at $O(\epsilon^3)$. The streamfunction $\psi^{(3)}$ at $O(\epsilon^3)$ consists of three terms:

$$\psi^{(3)} = \psi_{11}^{(3)}\cos\theta + \psi_{12}^{(3)}\sin\theta + \psi_{31}^{(3)}\cos 3\theta \tag{48}$$

only $\cos\theta$ and $\sin\theta$ components being relevant to the speed.

After lengthy but tedious algebra, the Navier-Stokes equations are collapsed into the following equation for $\psi_{11}^{(3)}$:

$$\frac{1}{r}\left(\frac{\partial\zeta^{(0)}}{\partial r}\psi_{11}^{(3)} + R_0 v^{(0)}\zeta_{11}^{(3)}\right) + R_0\dot{Z}^{(2)}\frac{\partial\zeta^{(0)}}{\partial r} = f(r)\,, \tag{49}$$

where

$$\zeta_{11}^{(3)} = \frac{1}{R_0}\Delta\psi_{11}^{(3)} - \frac{r}{R_0}\left\{\zeta_0^{(2)} + \frac{a}{2R_0}\tilde{\psi}_{21}^{(2)} + \frac{b}{8R_0^2}\left[a\tilde{\psi}_{11}^{(1)}\right]^2 + \frac{ra}{4R_0}\tilde{\psi}_{11}^{(1)}\right\}$$
$$-\frac{1}{R_0^2}\left(\frac{\partial\psi_0^{(2)}}{\partial r} + \frac{1}{2}\frac{\partial\psi_{21}^{(2)}}{\partial r} + \frac{\psi_{21}^{(2)}}{r}\right) + \frac{r}{4R_0^3}\left(3\frac{\partial\psi_{11}^{(1)}}{\partial r} + \frac{\psi_{11}^{(1)}}{r}\right) + \frac{3r^2}{4R_0^3}v^{(0)} \tag{50}$$

and

$$f(r) = \frac{1}{2R_0}\left(\frac{b}{r}\tilde{\psi}_{11}^{(1)} + a\right)v^{(0)}\tilde{\psi}_{21}^{(2)} + \frac{1}{4R_0^2}\left\{2a\tilde{\psi}_{21}^{(2)}\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r} + \frac{2b}{r}\left[\tilde{\psi}_{11}^{(1)}\right]^2\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\right.$$
$$\left.+\frac{1}{2r}\frac{\partial b}{\partial r}\left[\tilde{\psi}_{11}^{(1)}\right]^3 + \left(\frac{2a}{r} - \frac{3bv^{(0)}}{2}\right)\left[\tilde{\psi}_{11}^{(1)}\right]^2\right\}$$
$$+\left(\frac{\dot{Z}^{(0)}}{2R_0} + \frac{1}{R_0 r}\frac{\partial\psi_0^{(2)}}{\partial r} - \frac{rv^{(0)}}{2R_0^2}\right)a\tilde{\psi}_{11}^{(1)} + \zeta_0^{(2)}v^{(0)} - \frac{1}{r}\frac{\partial\zeta_0^{(2)}}{\partial r}\psi_{11}^{(1)}\,. \tag{51}$$

The boundary conditions are

$$\psi_{11}^{(3)} \propto r \quad \text{as}\ r\to 0\,, \tag{52}$$

and, from (19),

$$\psi_{11}^{(3)} \sim \frac{3}{2^7\pi R_0^2}\left[\log\left(\frac{8R_0}{\epsilon r}\right)-\frac{1}{3}\right]r^3 - \frac{\Gamma^{(2)}}{4\pi}\left[\log\left(\frac{8R_0}{\epsilon r}\right)-1\right]r$$
$$-\frac{d_0}{8R_0^2}\left[\log\left(\frac{8R_0}{\epsilon r}\right)-\frac{7}{4}\right]r$$
$$-\quad\left(\frac{1}{2\pi}\left\{\frac{1}{4}\left[2\pi\int_0^\infty r^3\zeta_0^{(2)}dr\right]+R_0\left[\pi\int_0^\infty r^2\zeta_{11}^{(3)}dr\right]+\frac{1}{4}\left[\pi\int_0^\infty r^3\zeta_{21}^{(2)}dr\right]\right\}\right.$$
$$\left.-\quad\frac{1}{8\pi R^2}\left\{-\frac{1}{2^6}\left[2\pi\int_0^\infty r^5\zeta^{(0)}dr\right]+\frac{R_0}{8}\left[\pi\int_0^\infty r^4\zeta_{11}^{(1)}dr\right]+\frac{R_0^2}{2}\left[\pi\int_0^\infty r^2\zeta_{21}^{(2)}dr\right]\right\}\right)\frac{1}{r}$$
$$\text{as } r\to\infty. \tag{53}$$

The last term of (53), being inversely proportional r, pertains to fixing the location of the core center with the accuracy of $O(\epsilon^3)$, but may be forgotten for determining the speed at the present order. To deduce $\dot{Z}^{(2)}$, we can skip the full solution of (49)–(53). It suffices to multiply (49) by r^2 and to integrate from 0 to some large value with respect to r. To simplify the expression, (40)–(47) is invoked. Taking the limit of $r\to\infty$, we eventually arrive at the desired formula:

$$\dot{Z}^{(2)} = \frac{\pi}{4R_0^3}\int_0^\infty\left[\frac{17}{8}rv^{(0)}-\frac{3}{R_0}\psi^{(0)}\right]r^3\zeta^{(0)}dr$$
$$-\frac{\pi}{R_0^2}\int_0^\infty\left[ra+\frac{b}{2}\tilde{\psi}_{11}^{(1)}\right]rv^{(0)}\tilde{\psi}_{21}^{(2)}dr-\frac{5\pi}{4R_0^3}B$$
$$+\frac{\pi}{8R_0^3}\int_0^\infty\left\{ra\left[r\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}+\tilde{\psi}_{11}^{(1)}\right]\tilde{\psi}_{11}^{(1)}-b\left[r\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}-\tilde{\psi}_{11}^{(1)}\right]\left[\tilde{\psi}_{11}^{(1)}\right]^2\right\}dr$$
$$+\frac{\pi}{2R_0^3}\int_0^\infty\left[rv^{(0)}-\frac{\psi^{(0)}}{R_0}-R_0\dot{Z}^{(0)}\right]r^2a\tilde{\psi}_{11}^{(1)}dr-\frac{3\Gamma^{(2)}}{8\pi R_0}$$
$$-\quad\frac{\pi}{R_0}\int_0^\infty\left[2rv^{(0)}+\frac{\psi^{(0)}}{R_0}\right]r\zeta_0^{(2)}dr+\frac{\pi}{R_0}\int_0^\infty\left[\frac{\partial\zeta_0^{(2)}}{\partial r}-\frac{a}{R_0}\frac{\partial\psi_0^{(2)}}{\partial r}\right]r\tilde{\psi}_{11}^{(1)}dr\,, \tag{54a}$$

where definitions (30) and (44) of a and b should be remembered, and

$$B=\lim_{r\to\infty}\left\{\int_0^\infty rv^{(0)}\tilde{\psi}_{11}^{(1)}dr+\frac{1}{4}\left(\int_0^\infty r[v^{(0)}]^2dr\right)r^2-\frac{d_0}{2\pi}\left[\log\left(\frac{8R_0}{\epsilon r}\right)-\frac{7}{10}\right]\right\}. \tag{54b}$$

The fact that (54a) includes the parameter d_0 brings out the contribution of the dipole distributed along the core-centerline to the induction velocity $O(\epsilon^3)$, which has so far gone unnoticed. This is traced back to the matching condition (53), essentially non-local in its nature. It cannot be overemphasised that the asymptotic formula (19) of the Biot-Savart integrable is indispensable to make the systematic evaluation of multi-pole induction feasible.

In order to get the value of $\dot{Z}^{(2)}$, there remains to numerically calculate $\psi_0^{(2)}$ and $\psi_{21}^{(2)}$. Fortunately, the explicit solution is at our hand for the Rankine vortex

(35). In this case,

$$B = \frac{3}{2^5\pi^2}\log\left(\frac{8R_0}{\epsilon}\right) - \frac{71}{15\cdot 2^5\pi^2}\,. \tag{55}$$

Noting that $a = -2\delta(r-1)$ and (44), the last four integrals of (54a) vanish and we are left with

$$\dot{Z}^{(2)} = -\frac{3}{2^5\pi^2 R_0^3}\left[\log\left(\frac{8R_0}{\epsilon}\right) - \frac{5}{4}\right], \tag{56}$$

in accordance with (1). Otherwise stated, (54a) is a generalisation of Dyson's result to an arbitrary distribution of leading-order vorticity in the presence or absence of viscosity.

The rest of this section concerns the third-order radial velocity $\dot{R}^{(2)}$. Equation of $\psi_{12}^{(3)}$ is reducible to

$$\begin{aligned}
&\frac{1}{r}\left(\frac{\partial\zeta^{(0)}}{\partial r}\psi_{12}^{(3)} + v^{(0)}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\psi_{12}^{(3)}\right)\right]\right) - R_0\dot{R}^{(2)}\frac{\partial\zeta^{(0)}}{\partial r} \\
&= R_0\left(-\frac{\partial\zeta_{11}^{(1)}}{\partial t} + \Delta\zeta_{11}^{(1)} + \frac{1}{R_0}\frac{\partial\zeta^{(0)}}{\partial r}\right),
\end{aligned} \tag{57}$$

subject to the matching condition

$$\psi_{12}^{(3)} \propto 1/r \quad \text{as } r\to\infty\,. \tag{58}$$

As before, we implement the integration of (57) with respect to r after multiplication by r^2. The diffusion equation (25) of $\zeta^{(0)}$ helps to simplify the result in such a way that

$$\frac{1}{R_0}\int_0^\infty r^2\frac{\partial\zeta^{(0)}}{\partial r}dr = -\frac{1}{2R_0}\frac{d}{dt}\int_0^\infty \zeta^{(0)}r^3\,dr\,, \tag{59}$$

and thus we obtain the speed of the origin $r=0$ of the local moving coordinates in the ρ-direction:

$$\dot{R}^{(2)} = \frac{2\pi}{R_0}\dot{d}_0\,. \tag{60}$$

With the aid of the initial condition $R^{(2)}(t=0)=0$, this can be integrated to give

$$R^{(2)}(t) = \frac{2\pi}{R_0}\left(d_0(t) - d_0(0)\right)\,. \tag{61}$$

It is noteworthy that (60) is consistent with the conservation law of the fluid impulse. Recall that the impulse is constant even in the presence of viscosity. Only the z-component P_z is nontrivial for the axisymmetric flow, giving, in dimensionless form,

$$\begin{aligned}
P_z &= \pi R_0^2 + \epsilon^2\left\{R_0^2\left[2\pi\int_0^\infty r\zeta_0^{(2)}dr\right] + 2R_0R^{(2)}\left[2\pi\int_0^\infty r\zeta^{(0)}dr\right] + \pi\int_0^\infty r^3\zeta^{(0)}dr\right. \\
&\quad \left. +2R_0\pi\int_0^\infty r^2\zeta_{11}^{(1)}dr\right\} + \cdots\,.
\end{aligned} \tag{62}$$

In the light of (38), the constancy of $O(\epsilon^2)$-term gives rise to (60). This observation implies that the first-order solution, combined with the impulse conservation, is sufficient to get $\dot{R}^{(2)}$ and therefore that we may skip the third-order solution $\psi_{12}^{(3)}$. Notice that the initial value of d_0 defined by (38) sets that of P_z up to second order. This manifests a remarkable aspect that our formulation of the initial-value problem rests upon the fundamental laws of conservation of both circulation and impulse.

Finally we illustrate how the vorticity distribution radially evolves starting from a delta-function core (26). In this case, $P_z = \pi R_0^2$ identically with $O(\epsilon^2)$ correction term being absent. The particular solution $\Psi_{11}^{(1)}$ given by (32) corresponds to the dipole field whose stagnation point is permanently sitting at $r = 0$ (Klein & Knio 1995). The evaluation of the behaviour of $\Psi_{11}^{(1)}$, at large values of r, is carried out with ease to yield

$$\Psi_{11}^{(1)} = \frac{r}{4\pi}\left\{\log r + \lim_{r\to\infty}\left(4\pi^2\int_0^r r'[v^{(0)}(r')]^2 dr' - \log r\right) + \frac{1}{2}\right\} + \frac{D_0}{r} + \cdots, \quad (63)$$

with

$$D_0 \cong 0.41225489 \times \frac{t}{2\pi}. \quad (64)$$

We reach the conclusion that, given initially a circular line vortex of radius R_0, the stagnation point $\rho_s(t)$ in the core drifts outward linearly in time owing to the action of viscosity:

$$\rho_s \cong R_0 + 0.41225489\nu t/R_0. \quad (65)$$

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